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NUMERICAL ANALYSIS OF CRACK DEVELOPMENT IN STRUCTURALLY NONUNIFORM COMPOSITES

E. A. Lankina UDC 539.2

The plane problem of an elastic unidirectional composite with a crack which grows from infinity at constant stress σ is considered here. The y axis coincides with the direction of reinforcement. If the dimensions of the binder H and the fiber h are small compared with the length of the crack, the macroscopic field far from the crack can be determined by the methods of continuum mechanics on the basis of integral equations from potential theory. Solution of the macroscopic problem in this formulation gives infinite growth of the stress upon approach to the crack margins. In the neighborhood of the crack margin, a formulation that takes into account the characteristic dimension of the real structure of the materials is necessary. Therefore, it is useful to break the problem down into two stages. In the first, the stress in a structureless composite is determined, i.e., the limiting case of a "smeared" structure is studied, when h, H → 0, h/H = const. Then a region around the crack margins in selected, and the stress determined for the smeared composite is used as a boundary condition on the boundary of this region. In the second stage, the interior of this region is described by equations that take into account the discrete structure of the composite, resulting in finite stresses. In this case the crack and the boundary of the selected region are considered to be an aggregate of fiber fractures and delaminations of binder. By using the strength conditions for fracture or delamination, the development of a crack is calculated, and parameter values are found at which cracks grow by fracture of the fibers or by delamination of the binder.

1. We denote displacement of the i-th fiber along the direction of reinforcement y by $u_i(y)$. Then the equation of equilibrium for the i-th fiber inside the composite is written as [17].

$$hH\frac{d^2u_i}{dy^2}+\beta^2(u_{i+1}-2u_i+u_{i-1})=0,\quad \beta^2=\mu/E \tag{1.1}$$

(μ, E are the moduli of elasticity for the binder and the fiber). The normal stress in the fiber and that tangential to the binder are computed from

$$\sigma_i=E\frac{\partial u_i}{\partial y},\quad \tau_i=\mu\frac{u_{i+1}-u_i}{H}. \tag{1.2}$$

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We take the limit of a uniform continuous unidirectional medium, for which h, H tend toward zero in such a way that their ratio is held constant, or, what is the same thing, the reinforcement coefficient $h/(h + H)$ is preserved. Dividing the second central difference $u_{i+1} - 2u_i + u_{i-1}$ in (1.1) by the square of the step-size in the horizontal coordinate $(H + h)^2$, we obtain the difference analog of the second derivative with respect to x. Multiplying (1.1) by the same and taking the limit, we have

$$\frac{1}{T}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad T=hH/\beta^2(H+h)^2. \tag{1.3}$$

Taking the limit in the second formula of (1.2) gives

$$\tau_{xy}=\mu\frac{\partial u}{\partial x}\frac{H+h}{H}. \tag{1.4}$$

The mean normal stress σ_{yy} for a continuum is obtained from the first expression in (1.2) if we consider that the concentration of fibers in the composite is equal to $h/(H + h)$:

$$\sigma_{yy}=E\frac{\partial u}{\partial y}\frac{h}{H+h}. \tag{1.5}$$

Expressions (1.4) and (1.5) can be considered as Hooke's law for a reinforced medium. Thus we represent Eq. (1.3) in the standard form for continuum mechanics:

$$\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\sigma_{yy}=0.$$

The load is given with the help of the condition at infinity

$$E\frac{h}{H+h}\frac{\partial u}{\partial y}=\sigma,\quad y\to\infty. \tag{1.6}$$

Let us derive the boundary condition at a crack with free margins. To do this, we construct the equilibrium equation for a small element which is formed by part of the crack ds and the projection of ds onto the coordinate axis (Fig. 1). Since the margins of the crack are stress-free, the equilibrium equation for such an element in the direction of the y axis has the form

$$\tau_{xy}ds\cos(n,\ x)+\sigma_{yy}ds\cos(n,\ y)=0$$

(n is the normal to the element ds). Using (1.4) and (1.5), we obtain

$$\mu\frac{H+h}{H}\frac{\partial u}{\partial x}\cos(n,x)+E\frac{h}{H+h}\frac{\partial u}{\partial y}\cos(n,y)=0.$$

We represent the elastic field as the sum of two fields: a uniform tensile stress σ of the plane without the crack, and a stress field around the crack with loaded margins. The displacements corresponding to the second (supplementary) problem is also denoted by u. It is easy to show that this displacement satisfies (1.3), the conditions at infinity (1.6) with a zero right-hand side, and the condition

$$\frac{1}{T}\cos(n,x)\frac{\partial u}{\partial x}+\cos(n,y)\frac{\partial u}{\partial y}=-\frac{\sigma}{E}\cos(n,y) \tag{1.7}$$

at both crack margins.

2. To solve problem (1.3), (1.7), we apply the methods of integral equations. If we carry out a substitution of the variable x, we arrive at Laplace's equation, while (1.7) is transformed into a derivative along the normal. On the basis of established properties of the harmonic potential for a double layer [2], it can be shown that the displacement

$$u(x,y)=\frac{1}{\pi}\int_s\varphi(s)\frac{(x-\xi)n_1(\xi)+(y-\eta)n_2(\xi)}{T(x-\xi)^2+(y-\eta)^2}ds\ \ ((\xi,\eta)\in s) \tag{2.1}$$

satisfies (1.3) if (x, y) does not belong to s, and during traversal of the line of integration, s undergoes a jump, equal to $2\varphi(x,y)/\sqrt{T}$ (n_1, n_2 are the components of the normal along the x and y axes). Therefore it is natural to seek the solution u(x, y) in the form of

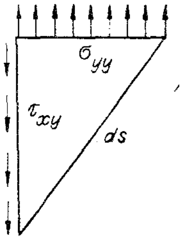

Fig. 1

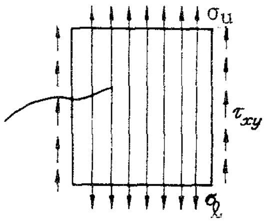

Fig. 2

potential (2.1), whose density φ is distributed along the crack line. We obtain the equation determining φ if we substitute (2.1) into (1.7), assuming that the point (x, y) lies outside the crack, and then taking the limit as (x, y) tends toward the crack. According to Liapunov's theorem [2], the result does not depend on the side from which (x, y) approaches the crack. To ensure the validity of these assertions, it is sufficient to require that the density and the direction cosines of the normal to the crack satisfy Holder's condition. To solve the problem, the crack line is approximated by a broken line, the ends of whose segments lie on the crack. The unknown density φ is approximated by a continuous piecewise linear function. The unknowns are the values of the density φ_i at the center of the i-th segment. Discretization of (1.7) leads to a linear system of equations for φ_i. The left-hand side of (1.7) (the derivative along the co-normal) is transformed into a finite sum of integrals of the form (2.1) along the broken line segments. In assembling the equation for the i-th segment, it is permissible to differentiate inside the integral in all intervals except the i-th interval. After differentiation, the value of the density is taken outside the integral over the broken line segment, and the remaining integrand is integrated analytically over this segment. As a result, the off-diagonal terms in the system of equations are obtained. After differentiating, the denominator in the integrand for the i-th interval has a second-order zero, which causes the integral to diverge. Therefore it is necessary to first compute the integral assuming that the point (x, y) lies on the perpendicular to the midpoint of the i-th segment, then take the derivative along the co-normal (1.7), and then take the limit of the resultant expression to the midpoint of the segment. This gives the diagonal term of the matrix of the linear system, which coincides with the finite part of the diverging integral in the sense of Hadamard [3]. The matrix elements obtained are quite cumbersome and will not be given here. However, for a straight-line crack, the expressions are simpler. We write out the system for a horizontal crack of length 2ℓ, $y = 0$, $-\ell < x < \ell$. We divide the crack into $(2\tilde{N} + 1)$ sections of identical length Δs. Then we have

$$\frac{1}{\pi}\sum_{j=-\tilde{N}}^{\tilde{N}} \frac{4}{T\Delta s}\,\frac{\varphi_j}{4(i-j)^2-1} = -\frac{\sigma}{E}. \tag{2.2}$$

The solution to system (2.2) was given in [1] (formula (3.4)). We write the displacement of the upper margin of the j-th segment of the crack as

$$u_j = \frac{1}{\sqrt{T}}\varphi_j = \frac{\sigma}{E}\,\frac{\Gamma(\tilde{N}-j+3/2)\,\Gamma(\tilde{N}+j+3/2)}{\Gamma(\tilde{N}-j+1)\,\Gamma(\tilde{N}+j+1)}\,\Delta s\sqrt{T}$$

(Γ is the gamma function). Using the Wallis formula (see [4])

$$\lim_{k\to\infty}\sqrt{k+1/2}\,\Gamma(k+1/2)/\Gamma(k+1) = 1$$

and the reduction formulae for the gamma function, we find

$$u_j \simeq \frac{\sigma}{E}\,\Delta s\sqrt{T}\sqrt{(\tilde{N}+1/2)^2-j^2}\ \text{for}\quad \tilde{N}\to\infty.$$

As $\tilde{N} \to \infty$, $\Delta s \to 0$ under the conditions $\Delta s\tilde{N} = \ell$, $\Delta s j = x$, we obtain the known solution for a straight-line slit:

$$u(x) = \frac{\sigma}{E}\sqrt{T}\sqrt{l^2-x^2}.$$

TABLE 1

Segment number	Exact solution	Numerical solution
1	0,348	0,439
2	0,582	0,638
3	0,726	0,770
4	0,827	0,865
5	0,899	0.935
6	0,949	0,983
7	0,982	1,014
8	0,998	1,029

Table 1 compares exact and approximate values of $uE/\sigma\ell\sqrt{T}$ for points on the crack margin with a division of the line of integration into 16 segments. (Because of the symmetric distribution of points with respect to the crack center, Table 1 gives the first 8 values for the displacement.) This example illustrates the convergence of the solution to the algebraic system and that of the integral equation.

3. Having solved the problem in the continuum formulation, we turn to a solution with structure taken into account. To do this, we select a rectangular region around the crack tip (Fig. 2) whose boundary is sufficiently far from the crack tip, so that the stress state on the boundary is determined with reasonable accuracy by the solution to the integral equation. After doing this, we solve (1.1) with boundary conditions on the boundaries of the selected region. The crack is then considered as a collection of fractures in fibers and delaminations of the binder. In [5] the solution of such a problem was reduced to an integro-algebraic system, the kernel and matrix of which were constructed taking into account the interaction of the elementary defects of the composite structure (fracture of the fibers and delamination). Here we will use a modification of this system, which accounts for the presence of tangential stresses at the vertical boundaries of the selected region, which are assumed to be delaminations of the binder in the unbounded composite. Considering the conditions at the boundaries of the selected region, we obtain, in accordance with [5], the displacement of the j-th fiber

$$
\begin{aligned}
u_j(\eta) = \sum_{m=1}^{N} \int_{\eta_{1m}}^{\eta_{2m}} \beta^2 G_j(\eta, \tau, j_m)\, \delta u_{jm}(\tau)\, d\tau + \\
+ \sum_{k=1}^{L} \sigma_k V_j(\eta, \eta_k, j_k) + \sum_{n=1}^{2} \int_{\eta_{1n}}^{\eta_{2n}} \tau_{xy}(\tau) K_j(\eta, \tau, j_n)\, d\tau + \\
+ \sum_{\gamma=1}^{M} \left(S_{\gamma 1} V_j(\eta, \eta_{\gamma 1}, j_\gamma) + S_{\gamma 2} V_j(\eta, \eta_{\gamma 2}, j_\gamma)\right).
\end{aligned}
$$

The integration interval $[\eta_{1m}, \eta_{2m}]$ corresponds to the limits of the m-th delamination of the binder;

$$
G_j(\eta, \tau, j_m) = \frac{(-1)^{j+j_m+1}}{M\beta} \sum_{k=1}^{M} g_k \sin\left(\frac{\pi k j_m}{M}\right) \sin\left(\frac{\pi k (j-1/2)}{M}\right) \exp(-2\beta\lambda_k |\eta - \tau|)
$$
$$
(g_k = 1\ (k = 1, \ldots, M-1),\ g_M = 1/2,\ \lambda_k = \cos(\pi k/2M_1))
$$

are the functions from [5] which describe the delamination of the binder; M is the total number of fibers; N the number of delaminations; L is the number of fractures; $\delta u_{jm} = u_{jm+1} - u_{jm}$;

$$
V_j(\eta, \eta_m, j_m) = (-1)^{j+j_m+1} \operatorname{sign}(\eta - \eta_m) \sum_{k=1}^{M} \frac{g_k}{M} \sin\left(\frac{\pi k (j-1/2)}{M}\right) \sin\left(\frac{\pi k (j_m - 1/2)}{M}\right) \exp(-2\beta\lambda_k |\eta - \eta_m|)
$$

are the functions from [1] which describe the fracture of the fibers. The limits of change for the ordinate $[\eta_{1n}, \eta_{2n}]$ determine the vertical dimension of the region being considered. The kernel

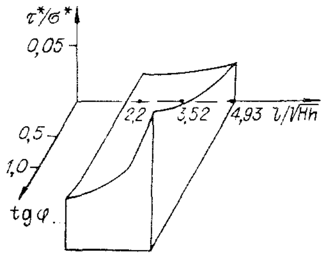

Fig. 3

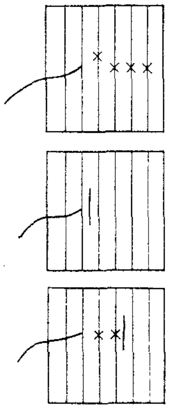

Fig. 4

$$K_j(\eta, \tau, j_n) = \frac{1}{8\pi\beta} \int_{-\pi}^{\pi} \frac{\exp(-2\beta|\sin(s/2)|) - \exp(-is(j-j_n))\exp(-2\beta|\eta-\tau||\sin(s/2)|)}{|\sin(s/2)|}$$

is the lumped-force solution to the problem in an infinite medium [6]. The unknowns of the system are δu_{jm}, σ_k, $S_{\gamma 1}$, and $S_{\gamma 2}$; the latter two are the strength of the dipole shift at the crack and at the horizontal boundaries during delamination. The system was solved on a computer, and then stresses were computed according to (1.2).

4. Using the described algorithm, the stability of a crack in a composite during the application of uniform tension at infinity was studied. It is known that, depending on the properties of the composite, a crack perpendicular to the fibers can either develop as a normal-failure crack (in which case growth is unstable), or it can give rise to a shear crack (which grows stably) [1]. Therefore it is of interest to study these possibilities for arbitrary crack orientation and to find the parameters of the composite structure which are conducive to delamination of the binder at the crack tip. To do this, let us consider a straight-line crack of length $2\ell/\sqrt{Hh} = 2.2, 3.52, 4.92$, which forms with angle φ to the horizontal ($\tan\varphi = 0, 0.5, 2$). The fracture of a fiber takes place when the condition $\sigma_j/\sigma^* = 1$ is met (σ_j is the maximum stress in the fibers, σ^* the fracture strength of the fibers). Delamination sets in when $\tau_j/\tau^* = 1$ (τ_j, τ^* are the analogous tangential stress values). The quantities σ_j, τ_j grow proportionally to the loading parameter σ; therefore the type of failure is determined by the ratio $\sigma_j\tau^*/\tau_j\sigma^*$. If this ratio is greater than 1, then fracture will set in first; if less than 1, delamination starts first. The ratio σ_j/τ_j is found from the solution to the problem. For $\varphi = 0$ it is given in [1], which makes it possible to pick out those parameters of the composite which separate the region of fiber fracture from that of delamination. In this paper, this ratio was determined numerically. Analysis shows that it depends on the parameters $\beta^2 = \mu/E$, h/H, $\ell/\sqrt{Hh}$, φ. Thus in combination with the parameter τ^*/σ^*, we have a five-dimensional space, each of whose points determines the mode of failure. Since the study of a five-dimensional space is too cumbersome, let us fix the parameters β^2 and h/H by taking values for the elastic constants from [7] and setting $h/H = 0.5$. Taking small variations of these parameters shows that the results depend very weakly on them. In the resultant three-dimensional space $\ell/\sqrt{HL}$, φ, τ^*/σ^*, we seek a surface separating the region of fracture from that of delamination. To this end, a series of problems was solved with $2\ell/\sqrt{Hh} = 2.2; 3.52; 4.92$; $\tan\varphi = 0; 0.5; 2$. The tensile stress σ was taken as equal to 1, since the results do not depend on this parameter. When determining the ratio σ_j/τ_j, the solution of the entire problem was used, i.e., the field of the uniform extension was added to the solution to the supplemental problem. The resultant surface is depicted in Fig. 3, and the points at which it is constructed are given in Table 2. The area lying above the surface corresponds to failure by fiber fracture; those below it to delamination of the binder. If the growth of a normal-failure crack takes place at small angles φ (i.e., the crack is nearly perpendicular to the fibers), then it grows without hindrance. But if the crack is sufficiently inclined to the fibers, then as it grows, it intersects the surface in Fig. 3 that separates the two failure modes, and fracture is replaced by stable delamination.

TABLE 2

$\tan\varphi$	$l/\sqrt{Hh}$ 2,2	3,52	4,92
	τ_k^*/σ_k^*		
0	0.0240	0.0244	0.0249
0.5	0.0259	0.0267	0.0278
2	0.0331	0.0473	0.1190

120

TABLE 3

$\tan\varphi$	$l/\sqrt{Hh}$ 2,2	3,52	4,92
	Probability of delamination		
0	50	60	60
0.5	60	60	80
2	80	100	80

5. In constructing Fig. 3, only the direction of stratification at the first instant of failure was examined, and subsequent crack development was not taken into account. This was done on the basis of plausibility arguments. In addition to this approach, it is possible to seek results on the basis of random variations in component strength. Below is described numerical experiments which make it possible to consider crack development in a composite with random strength. Let us assume that the strength is not constant, but is uniformly distributed about some mean value with a 5% spread. This case is modelled in the following way. Sweeping the entire computational region in the composite with a constant step-size of 0.4, we assign the strength at the nodes of the resultant mesh according to $\sigma^*(0.95 + 0.1a)$, $\tau^*\cdot(0.95 + 0.1b)$ (a, b are random numbers uniformly distributed over the interval (0.1) by a random-number generator on a computer). When determining the failure mode, just such local values for the strength were used. Then the points in Fig. 3 uniquely determine the failure mode only if they are sufficiently far from the surface separating the regions. Near this surface, the failure mode is determined only to some degree of probability, since the point can switch from one side of the surface to the other, due to random fluctuations. Since the relative scatter in the strength is small (±5%), the strength ratio is nearly symmetric with respect to τ^*/σ^*. Therefore it would seem that delamination and normal failure are equally probable. However, calculations indicate the predominance of delamination, especially in the region of large angles and lengths. The results are given in Table 3. The probabilities were computed for 10 experiments, for which the extent of crack motion was five fibers. With every advance of the crack, the change in problem geometry was taken into account and the stress state recomputed. Figure 4 shows typical possibilities for crack growth.

The predominance of delamination can be explained as follows. The first failure event can with equal probability be either fracture or delamination. Subsequently however, this symmetry is broken. The onset of delamination very sharply lowers the normal-stress concentrations in the fibers, and makes subsequent fracture unlikely. But if fiber fracture occurs, it does not hinder the occurrence of delamination at the next step, due to the random variation in strength. In addition, due to crack growth, it is possible to transfer the mapped points through the interface into the delamination region; as a consequence of this, subsequent fracture becomes less probable.

Thus, scatter in the strength properties of the composite elements makes crack stabilization possible, thereby increasing the load-bearing capacity of manufactured materials.

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